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A time-varying extremum-seeking control approach for discrete-time systems with application to model predictive control

Martin Guay *, **Ruud Beerens** **, **Henk Nijmeijer** **

* *Department of Chemical Engineering, Queen's University, Kingston, ON, Canada*
** *Department of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands*

Abstract: This paper considers the solution of a real-time optimization problem using adaptive extremum seeking control for a class of unknown discrete-time nonlinear systems. It is assumed that the equations describing the dynamics of the nonlinear system and the cost function to be minimized are unknown and that the objective function is measured. The main contribution of the paper is to formulate the extremum-seeking problem as a time-varying discrete-time estimation problem. The proposed approach is applied in the design of nonlinear model predictive control algorithms where the extremum-seeking controller is used to perform the real-time optimization of the MPC. A simulation study and an experimental study is presented that demonstrates the effectiveness of the proposed technique.

Keywords: Extremum-seeking control, Real-time optimization, Time-varying systems

1. INTRODUCTION

Extremum-seeking control (ESC) is a real-time optimization technique that is designed to track the optimum of a measured objective function. This approach, which dates back to the 1920s Leblanc [1922], provides an ingenious mechanism by which a system can be driven to the equilibrium of an unknown dynamical system that optimizes a measured variable of interest Tan et al. [2010]. Although the concept of ESC can be attractive for the solution of real-time optimization in practice, its development was hindered by the lack of stability results. In the late 1990s, Krstic and co-workers provided the first proof of convergence of a standard perturbation based extremum seeking scheme for a general class of nonlinear systems.

The vast majority of existing results on ESC have focussed on continuous-time systems. Although discrete-time systems can be treated in an essentially similar fashion, the application of gradient descent in a discrete-time setting requires some care. A discrete-time version of the standard ESC loop was studied in Ariyur and Krstic [2003] and Choi et al. [2002] where convergence results similar to continuous time systems are obtained. A similar algorithm was also proposed in Killingsworth and Krstic [2006] for the tuning of PID controllers in unknown dynamical systems using ESC. Discrete-time ESC subject to stochastic perturbations is studied in Manzie and Krstic [2009]. The use of approximate parameterizations of the unknown cost function using quadratic functions was recently proposed in Ryan and Speyer [2010]. An alternative ESC-like approach was proposed in Teel and Popovic [2001]. In this study, a trajectory based approach is used to analyze the properties of nonlinear optimization algorithms as dynamical systems. It is shown that properties of the nonlinear-optimization algorithms are suitable to assess the convergence of certain classes of ESC applied in a sampled-data approach. This approach was recently studied in the context of global sampling methods in Nesic et al. [2013] where trajectory based properties of nonlinear optimization methods are used to establish robust convergence. The main objectives with the trajectory based techniques is to analyze the properties of optimization algorithms assuming that they can converge to the true optimum using only the measurement of the objective function and possibly the constraints. In the context of ESC, one must either imply that the nonlinear optimization techniques do not rely on gradient information or, if they do, this gradient must be either measured or estimated. Some techniques such as Zhang and Ordóñez [2009] and Zhang and Ordóñez [2012] make use of sporadic gradient measurements in extremum seeking control. Other techniques Srinivasan [2007] go as far as requiring the existence of multiple (nearly) identical systems to enable the estimation of gradient information.

If a gradient based technique is considered, the estimation of the gradient must be addressed in some way. As highlighted above, this may be achieved by either parameterizing the cost function or by introducing a dither signal such that reliable gradient information can be extracted on average. If one considers a fixed parameterization of the cost, such as a quadratic cost function as in Ryan and Speyer [2010], (also Guay et al. [2004] and Cougnon et al. [2011] in continuous-time), the basic assumptions are that the parameters are sufficient slowly time-varying or constant and that the parameterization provides an accurate approximation of the unknown cost function. Since there is no way to establish in advance whether a given parameterization is suitable to achieve the optimization task, some estimation bias and loss in performance may result from the application of such techniques. In this paper, we provide an alternative discrete-time extremum-seeking technique which is based on the estimation of the gradient as a time-varying parameter. The main difference with existing technique is that the estimated time-varying gradient does not require any approximation or parameterization of the cost. The time-varying parameter estimation technique is used to remove the need for averaging system to establish the convergence of the extremum seeking controller to the unknown steady-state optimum of a measured output function. It also avoids the need to use the frequency of the dither signal as a singular perturbation parameter. The proposed ESC algorithm provides more freedom in the tuning of the ESC loop to achieve improvements in transient performance.

In this paper, we propose the application of the discrete-time ESC algorithm for the solution of nonlinear model predictive control (NMPC) problems. In particular, the ESC algorithm is applied to perform the real-time dynamic optimization in NMPC. The ESC-based NMPC is proposed as an alternative to fast MPC algorithms that have been proposed in the literature. Examples of fast algorithms include embedded online-optimization for MPC Jerez et al. [2013], tailored MPC algorithm Wang and Boyd [2010], explicit MPC algorithms Bemporad et al. [2011] and fast gradient approaches (Alamir [2013], DeHaan and Guay [2007]). The existing techniques for fast MPC employ various strategies to reduce computing time. Fast online optimization is achieved by either combining off-line and on-line computations, by designing highly tailored codes or by relaxing closed-loop performance requirements. One of the defining feature of most techniques is that they rely on a highly structured approach. The ESC approach proposed can be qualified as a highly unstructured approach that relies only on a parameterization of the predicted stage cost. The stage cost of the MPC is simply used as the measured cost to be optimized by the ESC.

The paper is organized as follows. A brief problem description is given in section 2. In section 3, the proposed ESC controller is presented for processes described by a static map. The application of the technique to a systems with unknown dynamics is presented in section 4. The ESC is applied in the design of a nonlinear MPC in section 5 for the swing up of a rotary inverted pendulum. This is followed by brief conclusions in section 6.

2. PROBLEM DESCRIPTION

Consider a nonlinear system

$$x_{k+1} = x_k + f(x_k, u_k) \tag{1}$$

$$y_k = h(x_k) \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the vector of state variables at the k^{th} time step, u_k is the vector of input variables at the k^{th} time step taking values in $\mathcal{U} \subset \mathbb{R}^p$ and $y_k \in \mathbb{R}$ is the variable to be minimized evaluated at the k^{th} time step. It is assumed that $f(x_k, u_k)$ is a smooth vector valued function of x_k and u_k and that $h(x_k)$ is a smooth function of x_k.

The objective of ESC is to steer the system to the equilibrium x^* and u^* that achieves the minimum value of $y(= h(x^*))$. The equilibrium (or steady-state) map is the n dimensional vector $\pi(u)$ which is such that:

$$f(\pi(u), u) = 0.$$

The equilibrium cost function is given by:

$$y = h(\pi(u)) = \ell(u) \tag{3}$$

Thus, at equilibrium, the problem is reduced to finding the minimizer u^* of $y = \ell(u^*)$.

Some basic assumptions are required to ensure that this problem is well-posed.

Assumption 1. The equilibrium cost (3) is such that

(1) $\frac{\partial \ell(u^*)}{\partial u} = 0$
(2) $\frac{\partial^2 \ell}{\partial u \partial u^T} > \alpha I, \ \forall u \in \mathcal{U}$.

where α is a strictly positive constant.

3. STATIC MAP

In this section, we consider the extremum-seeking problem for a static map:

$$y = \ell(u)$$

that satisfies Assumption 1.

In addition, the following assumptions are required.

Assumption 2. The static-map ℓ is such that

(1) $\|y\| \leq Y$
(2) $\left\|\frac{\partial \ell}{\partial u}\right\| \leq L_1$
(3) $\left\|\frac{\partial^2 \ell}{\partial u \partial u^T}\right\| \leq L_2$

$\forall u \in \mathcal{U}$ with positive constants $Y > 0$, $L_1 > 0$ and $L_2 > 0$.

In the development below, the minimization of y is performed in real-time. The input u is taken as a time-varying signal. That is,

$$y_k = \ell(u_k) \tag{4}$$

Consider the incremental variable, $\Delta y_k = y_{k+1} - y_k$, then it follows that:

$$\Delta y_k = \ell(u_{k+1}) - \ell(u_k)$$

Given that $\ell(u)$ is continuously differentiable, one can rewrite Δy_k as follows:

$$\Delta y_k = \int_0^1 \frac{\partial \ell(\lambda u_{k+1} + (1-\lambda)u_k)}{\partial u} d\lambda \Delta u_k,$$

where $\Delta u_k = u_{k+1} - u_k$. One can parameterize the quasi steady-state dynamics of the static map as:

$$\Delta y_k = \theta_k^T \Delta u_k \tag{5}$$

where the parameter

$$\theta_k = \int_0^1 \frac{\partial \ell(\lambda u_k + (1-\lambda)u_{k+1})}{\partial u} d\lambda$$

is a time-varying parameter. The strategy considered here is to develop a technique that can effectively estimate the time-varying behaviour of the parameters.

We must first make the following assumption concerning the input dynamics.

Assumption 3. The input signal u_k is such that

$u_k \in \mathcal{U}\ \forall k \geq 0.$

Remark 4. The Lyapunov argument presented below confirms that this can always be assumed. We note that one can enforce the boundedness of u via a projection algorithm. The inclusion of the projection algorithm would complicate the development below. However, it is entirely possible to do so in a formal way.

Finally, we note that by definition of θ_k, smoothness of $\ell(u)$ and boundedness of its gradient and hessian over all $u \in \mathcal{U}$, it follows that:

$$\|\theta_{k+1} - \theta_k\| \leq L_2\|u_{k+1} - u_k\| + L_2\|u_{k+2} - u_{k+1}.\| \quad (6)$$

The design of the extremum seeking routine is based on the local time-varying parameterization of the cost function (5). The first step consists in the estimation of the time-varying parameters θ. In the second step, we define a suitable controller that achieves the extremum-seeking task.

3.1 Parameter estimation

The model parameterization is given by:

$$\Delta y_k = \theta_k^T \Delta u_k = \phi_k \theta_k$$

where $\phi_k = [\Delta u_k]$ and $\theta_k = [\theta_{0,k},\, \theta_{1,k}]^T$. Let the estimator model for (5) be chosen as

$$\Delta \hat{y}_k = \hat{\theta}_k^T \Delta u_k = \phi_k^T \hat{\theta}_k \quad (7)$$

where $\hat{\theta}_k$ is the vector of parameter estimates. We define the output prediction error as $e_k = \Delta y_k - \Delta \hat{y}_k$.

The proposed parameter estimation update approach is given as follows.

Let $\Sigma \in \mathbb{R}^{n_\theta \times n_\theta}$ be generated by the following recursion

$$\Sigma_{k+1} = \alpha \Sigma_k + \phi_k \phi_k^T, \qquad \Sigma_0 = \alpha_1 I \succ 0, \quad (8)$$

where α_1 and α are strictly positive constants to be assigned. One considers the parameter update law given by

$$\Sigma_{k+1}^{-1} = \frac{1}{\alpha}\Sigma_k^{-1} - \frac{1}{\alpha^2}\Sigma_k^{-1}\phi_k \Upsilon_k \phi_k^T \Sigma_k^{-1}, \qquad \Sigma_0^{-1} = \frac{1}{\alpha} I \quad (9)$$

$$\bar{\hat{\theta}}_{k+1} = \text{Proj}\left[\hat{\theta}_k + \frac{1}{\alpha}\Sigma_k^{-1}\phi_k \Upsilon_k e_k, \Theta_0\right], \quad \hat{\theta}_0 = \theta^0 \in \Theta^0 \quad (10)$$

where $\Upsilon_k = (1 + \frac{1}{\alpha}\phi_k^T \Sigma_k^{-1} \phi_k)^{-1}$ and $\bar{\tilde{\theta}}_k = \bar{\hat{\theta}}_k - \theta_k$.

The operator Proj represents an orthogonal projection onto the surface of the uncertainty set applied to the parameter estimate. Following Goodwin and Sin [2013], the projection operator is designed such that

- $\hat{\theta}_{k+1} \in \Theta_0$
- $\bar{\tilde{\theta}}_{k+1}^T \Sigma_{k+1} \bar{\tilde{\theta}}_{k+1} \leq \tilde{\theta}_{k+1}^T \Sigma_{k+1} \tilde{\theta}_{k+1}$

One possible algorithm for the projection algorithm is as follows. Define the upper bound for $\|\theta\|$ $(= L_1)$. Let $R = \text{Chol}(\Sigma_{k+1})$ denote the Cholesky factor of Σ_{k+1}. Then we perform the following:

Algorithm 1. If $\|\hat{\theta}_{k+1}\| \geq L_1$ then

- Let $\delta = \frac{L_1 \hat{\theta}_{k+1}}{\|\hat{\theta}_{k+1}\|}$,
- Let $z_\rho = \sqrt{\delta^T \Sigma_{k+1} \delta}$,
- With $\rho = R\hat{\theta}_{k+1}$ define $\bar{\rho} = \frac{\rho z_\rho}{\|\rho\|}$,
- Let $\bar{\hat{\theta}}_{k+1} = R^{-1}\bar{\rho}$.

Otherwise,

- Let $\bar{\hat{\theta}}_{k+1} = \hat{\theta}_{k+1}$.

It is assumed that the trajectories of the system are such that the following condition is met.

Assumption 5. Goodwin and Sin [2013] There exists constants $\beta_T > 0$ and $T > 0$ such that

$$\frac{1}{T}\sum_{i=k}^{k+T-1} \phi_i \phi_i^T > \beta_T I, \ \forall k > T. \quad (11)$$

This requirement is a standard persistency of excitation condition that can be found in most references on adaptive control and adaptive estimation. The reader is referred to Goodwin and Sin [2013] for more details.

3.2 Controller design

We propose the following gradient descent controller:

$$u_{k+1} = u_k - k_g \hat{\theta}_k + d_k \quad (12)$$

where d_k is a bounded dither signal and k_g is the optimization gain, a positive constant to be assigned. It is assumed that $\|d_k\| \leq D\ \forall k \geq 0$ where D is a positive constant. Note that, since $\hat{\theta}_k$ and d_k are assumed to be bounded, then the controller (12) is such that $\|u_{k+1} - u_k\| \leq k_g L_1 + D$.

4. OPTIMIZATION IN UNKNOWN DYNAMICAL SYSTEMS

In this section, we consider the application of the optimization approach proposed in the last section to dynamical systems (1) with cost function (2). The approach proposed is to consider a two time-scale approach. In this approach, the dynamical system is assumed to operate at a faster time-scale with sampling rate $\epsilon\Delta t$. The steady-state optimization routine operates at the slow time scale with sampling time Δt. The proposed closed-loop extremum seeking controller takes the form:

$$\begin{aligned}
x_{k+1} &= x_k - f(x_k, u_k) && (13a)\\
y_k &= h(x_k) && (13b)\\
u_{k+1} &= u_k - \epsilon k_g \hat{\theta}_k + \epsilon d_k && (13c)\\
\Delta \hat{y}_k &= \phi_k^T \hat{\theta}_k && (13d)\\
\Sigma_{k+1}^{-1} &= \Sigma_k^{-1} + \epsilon\left(\frac{1}{\alpha} - 1\right)\Sigma_k^{-1} \\
&\quad - \frac{\epsilon}{\alpha^2}\Sigma_k^{-1}\phi_k \Upsilon_k \phi_k^T \Sigma_k^{-1} && (13e)\\
\bar{\hat{\theta}}_{k+1} &= \text{Proj}\left[\hat{\theta}_k + \frac{\epsilon}{\alpha}\Sigma_k^{-1}\phi_k \Upsilon_k e_k, \Theta_0\right] && (13f)
\end{aligned}$$

where $\Upsilon_k = (1 + \frac{1}{\alpha}\phi_k^T \Sigma_k^{-1} \phi_k)^{-1}$.

We let $\tilde{u}_k = u_k - u^*$ and define $\tilde{x}_k = x_k - \pi(\tilde{u}_k + u^*)$. The dynamics of $\tilde{x}_k$ are given by the recursion:

$$\tilde{x}_{k+1} = \tilde{x}_k - \pi(u_{k+1}) + \pi(u_k) + f(\tilde{x}_k + \pi(u_k), u_k).$$

At the faster time-scale, $\epsilon = 0$, it is assumed that the equilibrium $x_k = \pi(u_k)$ is locally asymptotically stable. As a result, the origin is a locally asymptotically stable equilibrium of the boundary layer dynamics:

$$\tilde{x}_{k+1} = \tilde{x}_k + f(\tilde{x}_k + \pi(u_k), u_k) \tag{14}$$

where u_k can be interpreted as a constant vector in $\mathcal{U}$.

Assumption 6. By the converse theorem of Lyapunov in discrete-time Khalil [1992], it follows that there exists a continuous function $V(\tilde{x})$ such that:

(1) there exists $\mathcal{K}$ functions α_1 and α_2 such that $\forall \tilde{x} \in \mathcal{D}$
$$\alpha_1(\|\tilde{x}\|) \leq V(\tilde{x}) \leq \alpha_2(\|\tilde{x}\|)$$
(2) a class $\mathcal{K}$ function α_3 such that
$$V(\tilde{x}_{k+1}) - V(\tilde{x}_k) \leq -\alpha_3(\|\tilde{x}\|).$$

Remark 7. It is also known that if there exists a continuous function V that satisfies the last two inequalities in Assumption 6 then there also exists a smooth function W that satisfies the same inequalities. Let us therefore assume that $V(x)$ is a smooth vector-valued function.

Next we state the main result of the paper.

Theorem 1. The discrete-time time-varying extremum seeking controller is such that there exists a small parameter ϵ^* such that $\forall \epsilon \in (0, \epsilon^*)$ for which the closed-loop system (13) subject to the assumptions 1-6 converges to a $\mathcal{O}(\epsilon)$ neighbourhood of the unknown optimum u^*.

Proof: See Appendix 7. ∎

5. ESC FOR MPC

In this section, we consider the use of ESC to perform the dynamic real-time optimization that arise in the implementation of a model predictive controller. A process model is used to predict the output of the system at future time instants. These outputs are usually obtained by minimizing a certain cost function depending on the state variables, where the future control sequence acts as a design variable. Then, a receding strategy is applied so that at each instant the horizon is displaced towards the future, which involves the application of the first control signal of the sequence calculated at each step Camacho and Bordons [2004]

Since MPC relies on minimization of a cost function, the computations are expensive and may be time consuming. In most cases, an iterative optimization method has to be used to calculate the optimal control sequence. This process can be relatively slow, which gives rise to a motivation to investigate an alternative to numerical optimization algorithms that are generally used to perform the minimization tasks. A possible alternative is to use Extremum Seeking Control as presented in the previous sections. Good performance of the ESC requires a fair amount of parameter tuning, which can be challenging and time consuming as well. This is in contrast to the use of numerical iterative optimization methods such as the algorithms available in the optimization toolbox of MATLAB, which require little or no tuning. These algorithms, however, may not be able to solve the optimization problem fast enough to perform the MPC dynamic optimization in real-time. An implementation of the ESC algorithm in MPC, on the other hand, is expected to be fast since it only requires a unique model prediction at each sampling instant. The question that arises is the extent to which ESC is a valuable alternative to an iterative optimization method in MPC. The results of this study are presented in the following sections.

5.1 Simulation study

To investigate the performance of the application of ESC to MPC, the swing-up and stabilizing control of a rotary inverted pendulum is discussed. The system considered is the Quanser rotary inverted pendulum system, which is driven by a rotary servo motor system. A rotary pendulum arm is mounted to an output gear attached to the motor. At the end of the arm, the pendulum is attached. The goal is to swing-up the pendulum to its unstable equilibrium point at the upright position and stabilize it. A nonlinear model of the system is derived by means of the Euler-Lagrange equations. They are given by:

$$\left(m_p r^2 + J_b\right)\ddot{\theta} + m_p r \ddot{\alpha} l_p \cos(\alpha) - m_p r \dot{\alpha}^2 l_p \sin(\alpha) = u \tag{15a}$$

$$m_p l_p \cos(\alpha)\ddot{\theta} r - m_p l_p \sin(\alpha)\dot{\alpha}\dot{\theta} r + m_p \ddot{\alpha} l_p^2 - m_p g l_p \sin(\alpha) = 0 \tag{15b}$$

where $m_p = 0.125$ [kg] is the mass of the rod, $l_p = 0.1714$ [m] is the distance from the arm to the centre of gravity of the rod, $J_b = 0.0038$ [kgm^2] is the inertia of the arm and gears, $r = 0.2159$ [m] is the length of the pendulum arm, α is the pendulum angle, θ is the arm angle and u is the input torque. Define the state vector as $x = \left[\theta \ \alpha \ \dot{\theta} \ \dot{\alpha}\right]^T$.

The control problem consists of two parts: the swing-up and stabilization. For the swing-up, a destabilizing PD-controller is designed, that destabilizes the stable equilibrium point corresponding to $\alpha = 0$ [rad] and eventually swings up the pendulum. The control law takes the form of a positive feedback controller, with an additional term δ whose value is to be determined by the ESC-MPC algorithm:

$$u = -\theta + (P\alpha + D\dot{\alpha}) + \delta \tag{16}$$

For this purpose, (13) is implemented in a Model Predictive control scheme which determines the sub-optimal future control inputs. The cost function to be evaluated is

$$J_k = \sum_{i=1}^{N} \left(x^T Q x + u^T R u\right) \tag{17}$$

where Q is a positive scaling matrix, R is a positive scaling constant and N is the prediction horizon length. A simulation is performed on the discretized system using the following numerical values: $N = 5$, $Q = diag([1\ 5\ 0\ 0])$, $R = 0.1$, $\alpha = 0.01$, $k_g = 0.005$, $\epsilon = 1$ $P = 0.1$, $D = 10^{-4}$ and the dither signal is:

$$d_k = 0.001 \sin(1.5^i k),\ i = 1 : N,\ k = 1 : M \qquad (18)$$

where M is the length of the control horizon to satisfy $|\alpha| \leq 10\pi/180$ [rad], i.e. the point where a stabilizing state feedback takes over to stabilize the pendulum at its upright position.

The results of the simulation study are presented in Figure 1. The ESC results are compared to the case where $\delta = 0$ where the optimal δ MPC control moves are computed using the MATLAB optimization toolbox (`fmincon`). The results indicate good controller performance for the ESC-MPC control architecture. A significant increase in performance is realized compared to the pure PD swing-up controller and the performance is as good as a full iterative optimization method is used.

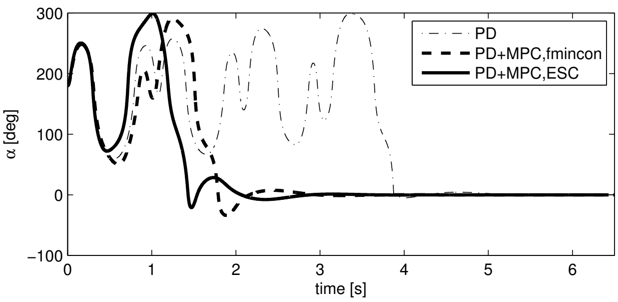

Fig. 1. Pendulum angle α as a function of time during swing-up and stabilization of the inverted pendulum. Simulation results

5.2 *Experimental study*

The inverted pendulum swing-up and stabilization control problem discussed in the previous section is applied to an experimental setup. Again the Quanser inverted pendulum setup is considered as described in the previous section. The calculated input torque is converted to an input voltage to the DC motor. The ESC-MPC control architecture, using (15) as the plant model for the prediction, is implemented and experiments are performed with the following numerical values: $\alpha = 0.05$, $k_g = 9 \cdot 10^{-5}$, $P = 0.08$, $D = 10^{-4}$ and the amplitude of the dither signal (18) is tuned to $5 \cdot 10^{-4}$. The swing-up using just the pure PD destabilizing controller is also investigated. The pendulum angle for both experiments are presented in Figure 2. The experimental results demonstrate that the ESC-MPC approach provides a significant increase in controller performance despite the fast nature of this process. The corresponding cost function profile is depicted in Figure 3. The cost function is effectively minimized and converges effectively to zero. Correspondingly, the output converges to the required setpoint. The corresponding control input δ determined by the ESC-MPC algorithm is shown in Figure 4.

Due to the high nonlinearity of this process and fast nature of its dynamics, it must be sampled at a fast rate (1000

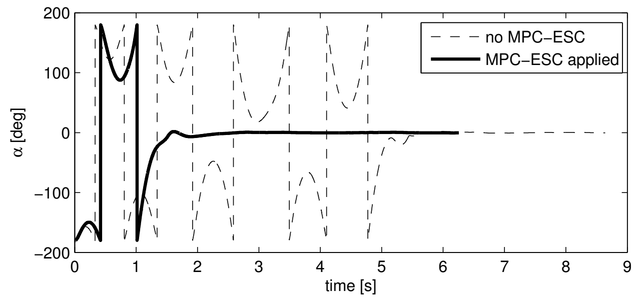

Fig. 2. Pendulum angle α as a function of time during swing-up and stabilization of the inverted pendulum. Experimental results.

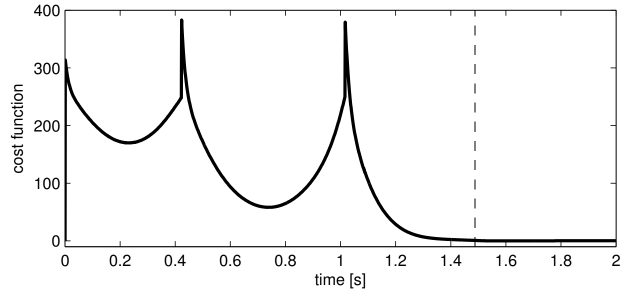

Fig. 3. Cost function profile. The swing-up controller is active on the left side of the dashed line, a state-feedback stabilizing controller is active on the right side.

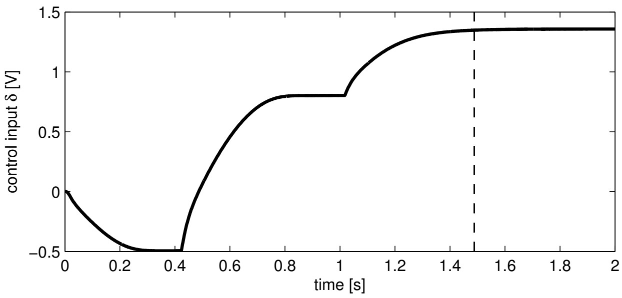

Fig. 4. Controller input δ determined by the MPC-ESC algorithm.

[Hz]). This limitation practically prevents the implementation of iterative nonlinear optimization methods for the implementation of MPC. The proposed ESC-MPC control architecture, in contrast, is capable of evaluating the system at this frequency and from the experimental results it can be concluded that ESC provides a valuable alternative for fast MPC computations. The implementation of ESC to MPC is straightforward and although the control architecture can be affected by poor tuning of several parameters, a fast controller resulting in high performance can be obtained using the proposed algorithm.

6. CONCLUSION

In this paper, an alternative ESC technique for unknown discrete-time dynamical systems was proposed. The technique is based on the time-varying estimation of the unknown gradient that is suitable for a very general and accurate parameterization of the unknown cost function.

A novel application of ESC for MPC is also considered. It is shown, both in simulation and experiments, that the proposed ESC technique can be used to compute MPC controls effectively. Future work will be devoted to the application of ESC for robust MPC in presence of uncertain process models.

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7. PROOF OF THEOREM 1

We provide a sketch of the proof. By the assumption on the $\tilde{x}$ dynamics, it can be assumed that there exists a function $V(x)$ such that

$$V(\tilde{x}_{k+1}) - V(\tilde{x}_k) \leq \epsilon L_V L_\pi (k_g L_\theta + D) - \alpha_3(\|\tilde{x}_k\|)$$

for constants L_V, L_π, L_θ, D and α_3. The reduced order dynamics consist of the extremum seeking controller and the parameter estimation scheme at the quasi steady-state $x_k = \pi(u_k)$.

$$\tilde{u}_{k+1} = \tilde{u}_k - \epsilon k_g \hat{\theta}_k + \epsilon d_k$$

$$\bar{\hat{\theta}}_{k+1} = \text{Proj}\left[\hat{\theta}_k + \frac{\epsilon}{\alpha}\Sigma_k^{-1}\phi_k(I + \frac{1}{\alpha}\phi_k^T\Sigma_k^{-1}\phi_k)^{-1}(e_k), \Theta_0\right].$$

The Lyapunov function $W_k = \tilde{\theta}_k^T \Sigma_k \tilde{\theta}_k$ can be shown to be such that:

$$W_{k+1} - W_k \leq 4\alpha_U\left(\epsilon^2 c^2 + \frac{\epsilon^2}{\alpha^2}L_f^2 L_h^2 (L_F^2\|\tilde{x}_k\|^2 + \epsilon^2 L_U^2)\right) - (1-\alpha-2\alpha^2)\alpha_L\tilde{\theta}_k^T\tilde{\theta}_k - \tilde{\theta}_k^{\,T}\phi_k Q_k \phi_k^T\tilde{\theta}_k.$$

for some constants L_f, L_h, L_F, L_U, α_U and α_L, Next, we pose the Lyapunov function $\mathcal{V}_k = V_k + W_k$. Its rate of change is given by:

$$\mathcal{V}_{k+1} - \mathcal{V}_k \leq -(1-\alpha-2\alpha^2)\alpha_L\|\tilde{\theta}_k\|^2 - \alpha_3(\|\tilde{x}_k\|) + \epsilon^2 K_1\|\tilde{x}_k\|^2 + \epsilon^2 K_2 + \epsilon K_3$$

where $K_1 = 4\alpha_u\frac{1}{\alpha^2}L_f^2L_h^2L_F^2$, $K_2 = 2\alpha_U c^2 + 2\frac{1}{\alpha^2}L_f^2L_h^2\epsilon^2L_U^2$ and $K_3 = L_V L_\pi(k_g L_1 + D)$. Finally, we pose the candidate Lyapunov function $U_k = \tilde{u}_k^T\tilde{u}_k$. It is easy to show that by combining U_k and $\mathcal{V}_k$, the following inequality is obtained:

$$U_{k+1} + \mathcal{V}_{k+1} - U_k - \mathcal{V}_k \leq - (1-\alpha-2\alpha^2-\epsilon\frac{k_g}{k_q})\alpha_L\|\tilde{\theta}_k\|^2 - \alpha_3(\|\tilde{x}_k\|) + \epsilon^2K_1\|\tilde{x}_k\|^2 - (2\epsilon k_g L_g - \epsilon k_g k_q - \epsilon k_d)\|\tilde{u}_k\|^2 + \epsilon^2K_2 + \epsilon K_3 + \epsilon K_4$$

where $K_4 = k_g(k_g+\epsilon)L_1 + (1+\epsilon+\frac{1}{k_d})D$. As a result, it follows that if, for some positive constants k_d and k_q, k_g, ϵ and α are chosen such that: $(1-\alpha-2\alpha^2-\epsilon\frac{k_g}{k_q}) > 0$, $(2\epsilon k_g L_g - \epsilon k_g k_q - \epsilon k_d) > 0$ and $-\alpha_3(\|\tilde{x}_k\|) + \epsilon^2 K_1\|\tilde{x}_k\|^2 < 0$, then $\|\tilde{u}_k\|^2$ enters an $\mathcal{O}(\epsilon(K_3+K_4))$ neighbourhood of the origin. And therefore, there exists an ϵ^* such that $\forall\epsilon \in (0,\epsilon^*)$, the input u_k enters an $\mathcal{O}(\epsilon(K_3 + K_4))$ neighbourhood of the unknown optimum u^*.